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Działania grup Coxetera na przestrzeniach ujemnie zakrzywionych

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Contents

Chapter 1

Introduction

1.1 Background and motivation

Coxeter groups form an important branch of geometric group theory and geometric topology. The classical motivation to study them is related to Lie groups and Tits buildings. This classical setting concerns mostly finite and affine Coxeter groups, whose behavior and classification is well understood. However, 'all other' (i.e. non-affine) Coxeter groups are certainly not to be dismissed. Indeed, their importance was recognized as soon as they provided useful examples in manifold theory, cohomology, and even in combinatorics of polyhedra (see [6] for a wide variety of such applications). The class of all non-affine Coxeter groups is thus very rich and it is usually difficult to establish their general properties. Nevertheless, the present dissertation attempts to make a step in this direction.

In the framework of geometric group theory it is natural to study geometric properties of spaces on which groups can act, and combinatorial properties of simplicial complexes related to these groups. In applications such spaces and complexes are often required to have some non-positive or even negative curvature properties. The reason for this curvature assumption is that there exists a well-developed theory of spaces satisfying curvature properties and, in addition, the existence of an appropriate action of a group on such a space has various geometric and algebraic consequences for the group itself. Theorem 1.4 may serve as an illustration of such consequences.

Every Coxeter group acts properly and cocompactly on its *Davis complex* which admits a CAT(0) metric, i.e. a metric that is globally non-positively curved. Moreover, it is known when this complex admits a CAT(κ) metric for some $\kappa < 0$, i.e. a globally negatively curved metric. A necessary and sufficient condition for this to happen has been established by Moussoung (see [6]), and it is equivalent to the Gromov-hyperbolicity of the Coxeter group under consideration. This condition says, roughly speaking, that there are no 'obvious' obstructions to hyperbolicity: there should be no Euclidean special subgroups of dimension at least 2 and no commuting infinite special subgroups. The details will be explained in Chapter 2. However, Coxeter groups that are not Gromov-hyperbolic themselves can still exhibit properties that are related to negative curvature. One of the strongest properties of this type is the existence of a non-elementary quotient that is a group acting properly and cocompactly on a CAT(κ) space for some $\kappa < 0$. Such a quotient is in particular a non-elementary Gromov-hyperbolic group. Any group G admitting such a quotient enjoys itself several algebraic properties: for example, it is SQ-universal (i.e. every countable group is a subgroup of some quotient of G, see [11]) and has uncountably many pairwise non-isomorphic torsion quotients (see [11]). This motivated the following question.

Conjecture 1.1 (Grigorchuk [11], Januszkiewicz [12])**.** *Every non-affine Coxeter group has a non-elementary Gromov-hyperbolic quotient.*

There is another application of the existence of such a quotient. Recall that there are two natural metrics on a group with a distinguished (usually finite) set of generators: the word metric, which measures the minimal number of factors in products of generators, and the bi-invariant metric, which measures the minimal number of factors in products of conjugates of generators. For a Coxeter group the bi-invariant metric is called *reflection length*. See Chapter 2 for details. Obviously the reflection length is bounded above by the length in the word metric. It turns out, however, that the reflection length is a bounded function on any affine Coxeter group, despite the fact that such a group is infinite and thus has unbounded word metric. In fact, McCammond and Petersen [13] demonstrated that for a Euclidean Coxeter group of dimension n the maximal value of the reflection length equals $2n$. This led them to the following question.

Conjecture 1.2 (McCammond, Petersen [13])**.** *The reflection length on any non-affine Coxeter group is an unbounded function.*

They proved that this conjecture is true for free Coxeter groups, i.e. free products of the form $\mathbb{Z}_2 * \mathbb{Z}_2 * \ldots * \mathbb{Z}_2$.

We will explain that if a group G has a non-elementary Gromov-hyperbolic quotient, then there exist elements of G of arbitrarily big bi-invariant length, and exactly this property accounts for the validity of Theorem 1.4. It is quite conceivable that there are more consequences of Gromov-hyperbolicity with applications yet to discover.

1.2 Main results

In this section we present and briefly discuss the main results of the thesis. Some of them, namely Theorems 1.3 and 1.4, have already appeared in print (K. Duszenko, *Reflection length in non-affine Coxeter groups*, [7]), but the thesis presents a revised and expanded version. The notions used in the formulations of the main results will be defined in Chapter 2.

Our first result proves Conjecture 1.1 for a special class of non-affine Coxeter groups.

Theorem 1.3. *Every minimal non-affine Coxeter group has a non-elementary quotient that acts properly and cocompactly on a CAT(κ) space for some $\kappa < 0$.*

In particular, this quotient is non-elementary Gromov-hyperbolic.

The proof of Theorem 1.3 relies on a modified and adapted version of filling constructions introduced by Bleiler and Hodgson [2] and by Mosher and Sageev [14]. The idea is to take a suitable subgroup $\overline{W}$ of a minimal non-affine Coxeter group W such that $\overline{W}$ is torsion-free and has finite index in W. Then $\overline{W}$ can be realized as the fundamental group of a hyperbolic manifold with cusps. Now take a compactification of this manifold by just adding one point compactifying each cusp. It turns out that the resulting compact space admits a metric of negative curvature, and evidently its Gromov-hyperbolic fundamental group $\overline{H}$ is a quotient of $\overline{W}$. Finally, it suffices to lift the surjection $\overline{W} \to \overline{H}$ to a surjection $W \to H$ (which is not at all an automatic thing as we explain at the end of Section 5.2).

Of course, some minimal non-affine Coxeter groups are Gromov-hyperbolic themselves, and then this construction is vacuous. But most of them contain Euclidean special subgroups, and our quotients reduce all of those to finite groups without reducing W itself too much, so that it is still non-elementary.

It is perhaps a bit surprising at first sight that even though Theorem 1.3 concerns only a certain family of non-affine Coxeter groups, it allows to prove Conjecture 1.2 in full generality. This is our second result.

Theorem 1.4. *The reflection length on any non-affine Coxeter group is an unbounded function.*

Here, one argues first that it is enough to prove Theorem 1.4 for minimal non-affine Coxeter groups. Then one applies Theorem 1.3 as well as results by Epstein and Fujiwara [9] about quasi-morphisms on Gromov-hyperbolic groups. Finally, one appeals to certain inequalities involving quasi-morphisms and bi-invariant metrics.

The last part of the dissertation is devoted to other cases when we are able to prove Conjecture 1.1. In the beginning we present a construction of quotients of all non-affine Coxeter groups that are conjecturally Gromov-hyperbolic. Then we describe two classes of groups for which the Gromov-hyperbolicity of the resulting quotients can be demonstrated.

The first class consists of groups admitting virtually free quotients. It turns out that these are exactly the Coxeter groups that have an infinite dihedral special subgroup not splitting as a direct factor. This statement is due to Januszkiewicz [12], however, the conference abstract [12] contains only a sketchy explanation that omits some important details. Therefore we take this opportunity to fill in the details, even though of course the fact itself cannot appear among the main results of the thesis.

The second class consists of 'small' non-affine Coxeter groups.

Theorem 1.5. *Every non-affine Coxeter group with* 4 *standard generators has a non-elementary Gromov-hyperbolic quotient.*

Many 4-generator Coxeter groups are Gromov-hyperbolic themselves. But this class also includes non-affine groups that are not Gromov-hyperbolic and not minimal non-affine in the sense of Theorem 1.3.

The proof of Theorem 1.5 will be based on the properties of a certain triangle of groups and mutual location of special subgroups in a Coxeter group. Results by Barnhill [1] concerning the latter will come into play here. It will also be quite evident from the proof why Conjecture 1.1 in full generality seems to be much more difficult.

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Chapter 2

Terminology and notation

This chapter is a very concise compilation of notions and notation used throughout the thesis. For a more systematic treatment of any of the below topics the reader is referred to the indicated textbooks. Some notions that are used only at one specific point of the dissertation are skipped here.

2.1 Coxeter groups

The references here are the book by Davis [6] or Bourbaki [3].

For a finite set S a *Coxeter matrix* is a square matrix $\{m_{st}\}_{s,t\in S}$ with entries subject to the following conditions: $m_{ss} = 1$ for every $s \in S$, and $m_{st} \in \{2, 3, 4, \dots, \infty\}$ for $s \neq t$. A *Coxeter system* corresponding to such a matrix is a pair (W, S), where W is a group defined by the presentation $\langle S | (st)^{m_{st}} = 1 \text{ for all } s, t \in S\rangle$, with the convention that the relation involving st is omitted whenever $m_{st} = \infty$. The group W is called a *Coxeter group*. One has to be careful: different Coxeter systems can give rise to Coxeter groups that are isomorphic as abstract groups. (In fact, the problem of describing exactly when this happens is still open.) For this reason, whenever we say 'a Coxeter group W', we will actually mean that there is a distinguished subset S of W such that the pair (W, S) is a Coxeter system, even if we deal with properties independent of S, such as being infinite, non-affine or Gromov-hyperbolic. Sometimes we will also say 'a Coxeter group (W, S)'.

For any Coxeter system (W, S) and any subset $T \subset S$, the subgroup W_T of W generated by T is called a *special subgroup*. It is a standard fact that (W_T, T) is itself a Coxeter system. Note that for any pair $s, t \in S$ such that $m_{st} = 2$ the relations $s^2 = t^2 = (st)^2 = 1$ imply that the generators s and t commute. In particular, if S is the disjoint union of subsets S_1 and S_2 such that $m_{ss'} = 2$ for all $s \in S_1$ and $s' \in S_2$, then the Coxeter group (W, S) decomposes as the direct product of the groups (W_{S_1}, S_1) and (W_{S_2}, S_2).

A Coxeter group is *finite* if it is finite as an abstract group. A finite Coxeter group (W, S) acts by isometries on a sphere of dimension $|S| - 1$. Moreover, the fundamental domain of this action is a simplex of maximal dimension and elements of S acts as reflections across codimension-1 faces of this simplex. A Coxeter group is *Euclidean* if it acts by isometries on a Euclidean space with the same properties as in the previous sentence. The classification of finite and Euclidean Coxeter groups is well-known. A Coxeter group is *affine* if it decomposes as a direct product of finite and Euclidean special subgroups, and *non-affine* otherwise.

Let (W, S) be a Coxeter group. A *reflection* is a conjugate of an element of S. Let $R = \{wsw^{-1} : s \in S, w \in W\}$ be the set of reflections. Then for any element $w \in W$ we define its *word length*

$$||w||_S = \min\{n : w = s_1 s_2 \dots s_n \text{ for some } s_i \in S\}$$

and *reflection length*

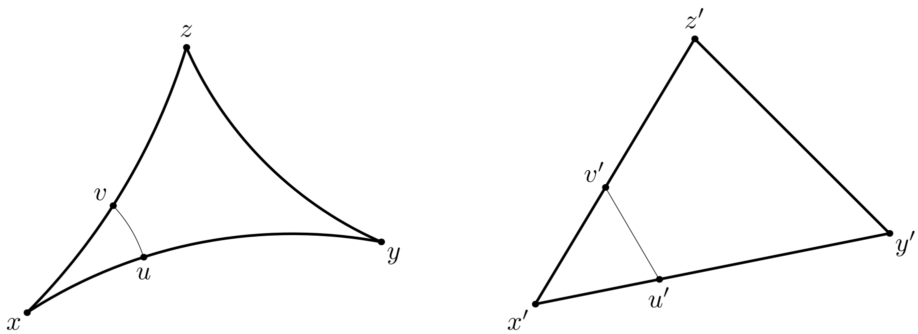
$$||w||_R = \min\{n : w = r_1 r_2 \dots r_n \text{ for some } r_i \in R\}.$$

2.2 Curvature and Gromov-hyperbolicity

The book by Bridson and Haefliger [4] is the standard reference for this topic.

A *geodesic segment* in a metric space is a subspace isometric to a closed segment on the real line. We say that a geodesic segment connects its two endpoints. A metric space is *geodesic* if any two points of this space can be connected by a geodesic segment. A subspace Y of a geodesic metric space X is *convex* if every geodesic segment in X with both endpoints in Y is entirely contained in Y. A *geodesic triangle* in a geodesic metric space consists of an ordered triple (x, y, z) of points ('vertices') and three geodesic segments $[x, y]$, $[y, z]$, $[z, x]$ ('sides').

A geodesic metric space X is CAT(0) if it satisfies the following 'no thick triangles' condition. For any geodesic triangle $\triangle$ in X with vertices x, y, z, let $\triangle'$ be a triangle with vertices x', y', z' in the Euclidean plane $\mathbb{E}^2$ such that $d_X(a, b) = d_{\mathbb{E}^2}(a', b')$ for any choice of symbols $a, b \in \{x, y, z\}$. $\triangle'$ is called a comparison triangle for $\triangle$. Then for any points $u \in [x, y]$, $v \in [x, z]$ and comparison points $u' \in [x', y']$, $v' \in [x', z']$, i.e. the points satisfying $d_X(x, u) = d_{\mathbb{E}^2}(x', u')$, $d_X(x, v) = d_{\mathbb{E}^2}(x', v')$, the following inequality should hold: $d_X(u, v) \leq d_{\mathbb{E}^2}(u', v')$. See the figure below. Intuitively, this means that geodesic triangles in X are not thicker than their comparison triangles in $\mathbb{E}^2$.



The same definition is used for the CAT(-1) condition, except that the Euclidean plane $\mathbb{E}^2$ should be replaced everywhere by the hyperbolic plane $\mathbb{H}^2$. More generally, for $\kappa < 0$ the CAT(κ) condition uses the rescaled hyperbolic plane $\mathbb{H}^2_\kappa$ as the comparison space; rescaling means that the Riemannian curvature of $\mathbb{H}^2_\kappa$ is equal to κ. There is also the CAT(1) condition, where the sphere $\mathbb{S}^2$ of radius 1 serves as the comparison space, and then one restricts the 'no thick triangles' property to geodesic triangles of perimeter not greater than 2π.

A geodesic metric space is non-positively curved (negatively curved) if every point has a convex open neighborhood which is CAT(0) (or CAT(κ) for some $\kappa < 0$, respectively) in the induced metric. In other words, such a metric space should satisfy the respective curvature condition locally. (Note here that the title of the dissertation might be slightly misleading: actually 'CAT(κ) for some $\kappa < 0$' should occur there instead of 'negatively curved', but this would make the title even more awkward.)

A geodesic metric space is δ-hyperbolic, where $\delta > 0$, if for every geodesic triangle any of its three sides is contained in the δ-neighborhood of the union of the other two sides. A space is *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta > 0$.

A finitely generated group is CAT(0) (Gromov-hyperbolic) if it acts properly and cocompactly by isometries on a CAT(0) (Gromov-hyperbolic, respectively) space. When the space is CAT(κ) for some $\kappa < 0$, we will call the corresponding group negatively curved. Such CAT(κ) spaces are Gromov-hyperbolic, and consequently negatively curved groups are Gromov-hyperbolic. (The converse is a famous open problem.) Note that this terminology might be inconsistent with many sources where negatively curved groups and Gromov-hyperbolic groups mean the same thing. Also, one usually says just 'hyperbolic' instead of 'Gromov-hyperbolic'. But then some confusion might arise what actually 'a hyperbolic space' is and what 'a hyperbolic Coxeter group' is (whether Gromov-hyperbolic or acting on some hyperbolic space $\mathbb{H}^n$). For this reason, we will refrain from using the phrase 'a hyperbolic group', and by a *hyperbolic space* we will always mean the space $\mathbb{H}^n$ for some positive integer n.

A group is *non-elementary* if it is infinite and not virtually cyclic.

2.3 Simplices of groups

The theory of complexes of groups in full generality is developed in [4]. Here we will only consider a special case sufficient for our purposes.

Suppose that G is a group acting by simplicial automorphisms on a connected and simply-connected simplicial complex X and that there is a simplex σ which is a fundamental domain for this action. Then this action can be encoded in the following way. To any face τ of σ (including σ itself) one assigns the stabilizer $\mathrm{Stab}_G(\tau)$ of this face. This is a *simplex of groups*: an assignment of a *local group* G_τ to every face $\tau \subset \sigma$ such that for any pair of faces $\tau_1 \subset \tau_2$ there is an injective homomorphism $\varphi_{\tau_2\tau_1} : G_{\tau_2} \to G_{\tau_1}$ and, moreover, the compatibility condition $\varphi_{\tau_2\tau_1} \circ \varphi_{\tau_3\tau_2} = \varphi_{\tau_3\tau_1}$ holds for any triple of faces $\tau_1 \subset \tau_2 \subset \tau_3$.

A simplex of groups is called *developable* if it arises from an action of a group G on a simplicial complex as described in the beginning of the previous paragraph. The uniquely determined group G is called *the fundamental group of the simplex of groups* and the recovered simplicial complex is called the *development*. We have the following algebraic critetion: a simplex of groups is developable if and only if there exists a group H and a family of injective homomorphisms $\{\psi_\tau : G_\tau \to H\}_{\tau\subset\sigma}$ satisfying the compatibility condition $\psi_{\tau_1} \circ \varphi_{\tau_2\tau_1} = \psi_{\tau_2}$ for any pair of faces $\tau_1 \subset \tau_2$. We usually say that the family $\{\psi_\tau\}_{\tau\subset\sigma}$ is a morphism injective on the local groups.

The simplices of X are in bijective correspondence with cosets of local groups in G, and the stabilizers of these simplices are conjugates of local groups.

Whenever presentations of the local groups $\{G_\tau\}_{\tau\subset\sigma}$ are known, one can obtain a presentation for the fundamental group of the corresponding simplex of groups in the following way: one first takes the free product of all the groups G_τ and then one adds relations of the form $\varphi_{\tau_2\tau_1}(x) = x$ for any pair of faces $\tau_1 \subset \tau_2$ and any element $x \in G_{\tau_2}$ (obviously one can take only elements x belonging to a set of generators of G_{τ_2}).

Finally, we describe an example that will be later generalized in Chapter 5.

Let (W, S) be a Coxeter system and let $S' \subset S$ be a subset with at least 3 elements. Consider a simplex σ with vertex set S'. A face τ of σ can be identified with the subset of S' equal to the set of all vertices of τ. The assignment of local groups is as follows: to the face of σ corresponding to a subset $T \subset S'$ we assign the special subgroup $W_{S\setminus T}$. The maps $\varphi_{\tau_2\tau_1}$ are just inclusions. This simplex of groups clearly admits a morphism to W injective on the local groups, and so it is developable. Putting together presentations of local Coxeter groups and the method of producing a presentation of the fundamental group one readily sees that the fundamental group of this simplex of groups is equal to W. (This is usually not true if $|S'| = 2$. Indeed, if $S' = \{s, t\}$, then the fundamental group is $W_{S\setminus\{s\}} *_{W_{S\setminus\{s,t\}}} W_{S\setminus\{t\}}$ and the relation $(st)^{m_{st}} = 1$ is missing unless $m_{st} = \infty$.)

For $S' = S$ the simplicial complex $\Sigma(W, S)$ obtained as the development of the above simplex of groups is called the *Coxeter complex of* (W, S). The maximal simplices (*chambers*) of $\Sigma(W, S)$ are of dimension $|S| - 1$, the W-action on $\Sigma(W, S)$ is simply transitive on the set of chambers, and the stabilizer of a simplex of $\Sigma(W, S)$ corresponding to a subset $T \subset S$ is a conjugate of the special subgroup $W_{S\setminus T}$.

Chapter 3

Quotients of minimal non-affine Coxeter groups

This chapter is devoted to the proof of Theorem 1.3. The reader is hereby warned that the notation in this chapter is developed gradually and, once introduced, will be valid throughout the whole chapter.

3.1 Actions on hyperbolic spaces

The first step is to study actions of minimal non-affine Coxeter groups on hyperbolic spaces. It turns out that every such group is a lattice in some hyperbolic space and acts on this space as a reflection group.

Proposition 3.1. *Every minimal non-affine Coxeter group (W, S) can be faithfully represented as a discrete group acting by isometries on the hyperbolic space $\mathbb{H}^n$, where $n = |S| - 1$. Moreover, the elements of S act as reflections across codimension-1 faces of some n-dimensional simplex σ which is contained in $\mathbb{H}^n$ with the possible exception that some vertices of σ might be ideal (i.e. they might lie on the boundary $\partial\mathbb{H}^n$).*

This fact is apparently known to the experts, although we were unable to find this exact statement in the literature — which is surprising in view of the analogy to 'classical reflection groups' acting on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. One can, however, deduce Proposition 3.1 without much effort from either ([15], Theorems 7.2.5 and 7.3.2 and the discussion after Theorem 7.3.2) or ([3], Exercises 12b, 12c, and 13 for Section V.4).

It follows from the proof that if (W, S) is a minimal non-affine Coxeter group, then for any $s \in S$ the special subgroup $W_{S\setminus\{s\}}$ is either finite or Euclidean, i.e. if it is infinite, then it does not decompose as a direct product of proper special subgroups.

Now fix a minimal non-affine Coxeter group W. By Proposition 3.1 we can, and will, consider W as a group of isometries of the hyperbolic space $\mathbb{H}^n$. Like any finitely generated linear group, W is residually finite and has a torsion-free normal subgroup of finite index (see for instance [15], Section 7.6).

Let W' be any torsion-free normal subgroup of W of finite index. Such a group W' acts freely on $\mathbb{H}^n$ and the quotient $\mathbb{H}^n/W'$ is a complete n-dimensional hyperbolic manifold, that is, a manifold with constant sectional curvature equal to -1. If all vertices of σ lie in $\mathbb{H}^n$, then this manifold is compact. However, if some vertices of σ are ideal, then this manifold is not compact, has finite volume and has finitely many *cusps*, each having a horoball neighborhood homeomorphic to the product of an $(n-1)$-dimensional compact flat manifold and a half-line. This cusp is just the quotient of a horoball in $\mathbb{H}^n$ — centered at some W-translate of an ideal vertex of σ — by the group W' which acts properly and cocompactly on the boundary of the horoball, isometric to the Euclidean space $\mathbb{E}^{n-1}$ if equipped with the Riemannian metric induced from the Riemannian metric of $\mathbb{H}^n$. (See [15], Chapters 9-11 for a systematic treatment of hyperbolic manifolds of finite volume.) Moreover, the Riemannian metric ds of a cusp can be described explicitly: for a cusp $T \times (-\infty, 0]$, where T is a compact flat manifold with Riemannian metric ds_T and the second coordinate is denoted by r, we have the following equality:

$$ds^2 = e^{2r} ds_T^2 + dr^2.$$

The basic idea of the proof of Theorem 1.3 is very similar to the procedure developed by Mosher and Sageev in [14], which produces Gromov-hyperbolic quotients of fundamental groups of hyperbolic manifolds: one should compactify each cusp by a point and prove the existence of a negatively curved metric on the resulting space. The difference is that we actually wish to construct a quotient of the group W, and not just of some subgroup W' of finite index. For this reason the results of [14] cannot be applied directly. We need to review and adapt these methods to our situation, where we additionally have a group action on a hyperbolic manifold that should be preserved by the construction. Nevertheless we will just refer to an appropriate part of [14] or [2] wherever possible.

We start with the existence of a 'sufficiently deep' subgroup of W of finite index such that the flat manifolds associated to the cusps do not have short closed geodesics.

Proposition 3.2. *There exists a torsion-free normal subgroup $\overline{W} \lhd W$ of finite index, with the following property: One can remove open horoball neighborhoods of all cusps of the hyperbolic manifold $\mathbb{H}^n/\overline{W}$ to obtain a manifold M with boundary composed of compact flat manifolds, such that:*

1. *Any closed geodesic in a component of ∂M has length greater than 2π.*
2. *The isometric action of the finite group $F = W/\overline{W}$ on the manifold $\mathbb{H}^n/\overline{W}$ restricts to an isometric action of F on M preserving ∂M.*

Proof. Observe first that the collection of simplices $\{w(\sigma) : w \in W\}$ forms a tessellation of $\mathbb{H}^n$ with some ideal vertices, namely the images of the ideal vertices of σ under the action of W.

Let $E(S)$ be the set of all elements $s \in S$ such that $S \setminus \{s\}$ generates a Euclidean special subgroup V_s of W. Consider an ideal vertex of the simplex σ. It corresponds to a generator $s \in E(S)$ such that the group V_s is the stabilizer of this ideal vertex. Let B_s be a small open horoball centered at this vertex and let $\mathbb{E}_s = \partial B_s$ be the Euclidean boundary of this horoball. Then $\tau_s = \mathbb{E}_s \cap \sigma$ is a Euclidean simplex fundamental for the action of the Coxeter group V_s on the Euclidean space $\mathbb{E}_s$.

We now have a family $\mathcal{B}_0 = \{B_s\}_{s\in E(S)}$ of pairwise disjoint horoballs. Therefore

$$\mathcal{B} = \{wB : w \in W, B \in \mathcal{B}_0\}$$

is a W-invariant collection of horoballs in $\mathbb{H}^n$ centered at the ideal vertices of the tessellation. We claim that $\mathcal{B}$ is a collection of pairwise disjoint horoballs.

The facts that σ is a fundamental domain for the action of W on $\mathbb{H}^n$ and that the collection $\mathcal{B}$ is W-invariant imply that if there exist two distinct intersecting horoballs in $\mathcal{B}$, then there also exist two distinct horoballs in $\mathcal{B}$ having a common point belonging to the simplex σ. Our claim will thus follow if we show that every horoball in $\mathcal{B}$ that intersects the simplex σ is in fact an element of the (pairwise disjoint) collection $\mathcal{B}_0$. But any horoball $B \in \mathcal{B}$ is of the form $w(B_s)$ for some $w \in W$ and $s \in E(S)$. Moreover, τ_s is a fundamental domain for the action of V_s on $\mathbb{E}_s$ and hence the intersection $B_s \cap \sigma$ is a fundamental domain for the action of V_s on B_s. It follows that the horoball B_s intersects the simplex $w'(\sigma)$ if and only if $w' \in V_s$. Since $w(B_s) \cap \sigma \neq \emptyset$, we see that $B_s \cap w^{-1}(\sigma) \neq \emptyset$. Consequently, $w \in V_s$ and $B = w(B_s) = B_s \in \mathcal{B}_0$, which proves the claim.

Also, due to W-invariance of the collection $\mathcal{B}$ the group W acts by isometries on the complement $\mathbb{H}^n \setminus \bigcup \mathcal{B}$.

Now, for every generator $s \in E(S)$ we define a finite set $A_s \subset V_s$ as follows:

$$A_s = \{v \in V_s \setminus \{1\} : d_{\mathbb{E}_s}(\tau_s, v(\tau_s)) \leq 2\pi\}.$$

Since W is residually finite and has a torsion-free subgroup of finite index, we can find a torsion-free normal subgroup $\overline{W} \triangleleft W$ of finite index disjoint with each of the sets A_s.

We claim that $\overline{W}$ is our desired group.

The union $\bigcup \mathcal{B}$ is W-invariant, and so is its boundary composed of horospheres. Hence we can define the required manifold M as the quotient of $\mathbb{H}^n \setminus \bigcup \mathcal{B}$ under the action of $\overline{W}$. The isometric action of W on $\mathbb{H}^n \setminus \bigcup \mathcal{B}$ descends to an isometric action of the finite quotient group $F = W/\overline{W}$ on the compact quotient manifold M, and clearly ∂M is preserved under that action. This proves (2).

It remains to show (1). A closed geodesic in a component of ∂M can be lifted to a geodesic segment in the boundary of a horoball from $\mathcal{B}$ with endpoints belonging to the same $\overline{W}$-orbit. We would like to prove that the Euclidean distance between these endpoints is greater than 2π. Therefore it suffices to verify that for each point p lying on the boundary ∂B of some horoball $B \in \mathcal{B}$ and for any non-trivial element $w \in \overline{W}$ such that $w(p) \in \partial B$, the Euclidean distance between p and $w(p)$ in ∂B satisfies $d(p, w(p)) > 2\pi$. But there exists an element $u \in W$ such that $u(B) = B_s$ for some $s \in E(S)$. Then the point $u(p)$ lies on the horosphere $\mathbb{E}_s = \partial B_s$. The simplex τ_s is a fundamental domain for the action of the Euclidean group V_s on $\mathbb{E}_s$. Hence after multiplying u on the left by a suitable element of V_s we may assume that $u(B) = B_s$ and $u(p) = q \in \tau_s$. Therefore the Euclidean distances in respective horospheres satisfy the equalities

$$d(p, w(p)) = d(u^{-1}(q), w(u^{-1}(q))) = d(q, u(w(u^{-1}(q)))) = d(q, w^*(q)),$$

where $w^* = uwu^{-1} \in \overline{W}$ as $\overline{W}$ is normal in W. Now w^* is non-trivial and $w^* \notin A_s$ for all $s \in E(S)$. Thus the definition of the sets A_s implies that $d(q, w^*(q)) > 2\pi$. □

From now on fix a subgroup $\overline{W}$ as in Proposition 3.2. The symbols F and M also retain their meaning.

3.2 Virtual quotients

We are going to construct a non-elementary Gromov-hyperbolic quotient of the group $\overline{W}$, or more precisely, a negatively curved space N whose fundamental group is a non-elementary quotient of $\overline{W}$. To this end we review some methods of [2] and some portions of the proof of Negative Curvature Theorem in [14], and modify these methods wherever necessary. One of the principal reasons for such an approach is that we have an action of the finite group F on the manifold M, which we would like to transfer to an action of F on N. This will be crucial for lifting the homomorphism $\overline{W} \to \pi_1(N)$ to a homomorphism defined on W.

Let $\mathcal{T}$ be the collection of connected components of ∂M. Recall that every manifold $T \in \mathcal{T}$ with the Riemannian metric induced from M is flat, i.e. it is a quotient of the Euclidean space $\mathbb{E}^{n-1}$ with the standard metric by a discrete cocompact group of isometries. But clearly T is not convex in $\mathbb{H}^n/\overline{W}$.

For each element $T \in \mathcal{T}$ we choose a number $r_T \in (-L_T/2\pi, -1)$, where L_T is the length of the shortest closed geodesic in T. In addition, we impose the following compatibility condition: we choose the numbers r_T in such a way that whenever two different elements $T_1, T_2 \in \mathcal{T}$ are isometric manifolds, we have the equality $r_{T_1} = r_{T_2}$. (In fact, all of the numbers $\{r_T\}_{T\in\mathcal{T}}$ may be just chosen to be equal.)

Then for each $T \in \mathcal{T}$ consider the product $T \times [r_T, 0]$ and collapse the subset $T \times \{r_T\}$ to a point p_T, thus obtaining a cone $C(T)$ over the manifold T. Now attach the cone $C(T)$ to the manifold M by identifying each point $(x, 0) \in T \times \{0\} \subset C(T)$ with the point $x \in T \subset M$. As T is a boundary component of M, it has a collar neighborhood in M, which can be thought of as the product $T \times [0, \epsilon)$ for some small $\epsilon > 0$. Finally, let N be the space obtained from M by attaching the cones $C(T)$ for all $T \in \mathcal{T}$. Note that N is an n-dimensional pseudomanifold, and in fact a manifold except at the points p_T.

Our goal now is to put a negatively curved metric on N so that the isometric action of F on M extends to an isometric action on N. On the manifold $N \setminus \{p_T : T \in \mathcal{T}\}$ this will actually be a Riemannian metric with the following property: for any component $T \in \mathcal{T}$ the metric on the subset $T \times (r_T, \epsilon)$ will be of the form

$$(*) \qquad ds^2 = (f_T(r))^2 ds_T^2 + dr^2,$$

where $f_T : (r_T, \epsilon) \to (0, \infty)$ is a certain function, ds_T is the flat Riemannian metric of the manifold T and r is the coordinate of the second factor of the product $T \times (r_T, \epsilon)$.

Let ds be the Riemannian hyperbolic metric on M, induced from the Riemannian metric on $\mathbb{H}^n$. Pick a boundary component $T \in \mathcal{T}$ and let ds_T be the restriction of ds to T; this is a flat metric on T. Then for some small $\epsilon > 0$ the metric ds on the collar neighborhood $T \times [0, \epsilon) \subset M$ is given by the formula

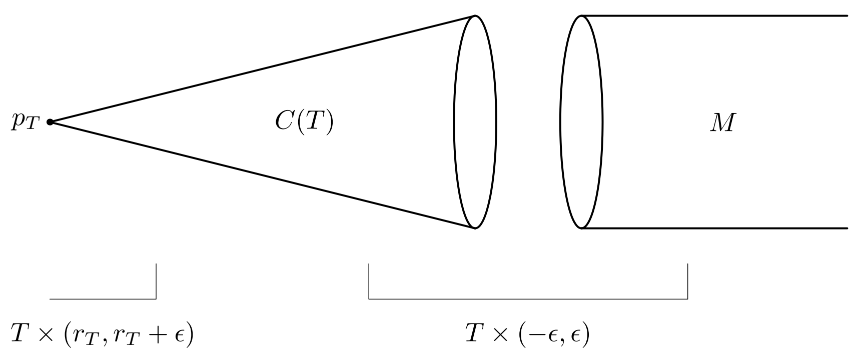
$$ds^2 = e^{2r} ds_T^2 + dr^2.$$

Extend this formula for a Riemannian metric to the set $T \times (-\epsilon, \epsilon) \subset N$. Now define a Riemannian metric on $T \times (r_T, r_T + \epsilon)$ by

$$ds^2 = \left(\frac{2\pi}{L_T} \sinh(r - r_T) \right)^2 ds_T^2 + dr^2.$$

The metric is now defined on the set $T \times [(r_T, r_T + \epsilon) \cup (-\epsilon, \epsilon)]$ — see the figure below — and it satisfies the relation $(*)$, where

$$f_T(r) = \begin{cases} \frac{2\pi}{L_T} \sinh(r - r_T) & \text{for } r \in (r_T, r_T + \epsilon), \\ e^r & \text{for } r \in (-\epsilon, \epsilon). \end{cases}$$



Note that the function f_T has not yet been defined on the whole interval (r_T, ϵ). On the other hand, the metric ds defined so far coincides with the original metric on M on the intersection of $T \times [(r_T, r_T + \epsilon) \cup (-\epsilon, \epsilon)]$ with M.

We now extend the function f_T defined so far to a smooth positive increasing convex function on the whole interval (r_T, ϵ). It is explained in ([2], Lemma 10 and Figure 1) that such an extension exists. Finally, the Riemannian metric ds on the set $T \times (r_T, \epsilon)$ is defined as in $(*)$ and, of course, the Riemannian metric on the interior of M is not modified at all, so that it has constant sectional curvature equal to -1 there.

To extend the above metric to the points p_T it suffices to note that, by the definition of the metric ds on the set $T \times (r_T, r_T + \epsilon)$, the diameter of $T \times \{r\}$ tends to zero when $r \to r_T$. Thus we may simply take the completion of the already defined metric on $N \setminus \{p_T : T \in \mathcal{T}\}$. In this completed metric N is a compact geodesic space.

Proposition 3.3. *The above construction has the following properties:*

1. *The compact space N is negatively curved.*
2. *Any isometry $\rho : T_1 \to T_2$ between (not necessarily distinct) boundary components $T_1, T_2 \in \mathcal{T}$ of M extends to an isometry $\widehat{\rho} : C(T_1) \to C(T_2)$. Consequently, the isometric action of F on M extends to an isometric action on N.*

Proof. Statement (1) consists of two independent parts: proving that the Riemannian metric defined by the formula $(*)$ on every set $T \times (r_T, \epsilon)$ has negative sectional curvature, and showing that every point p_T has a convex open neighborhood which is a CAT(-1) space.

The first part, which in fact does not require any assumptions about the length of closed geodesics in ∂M, follows easily from the computations done in ([2], Lemma 10). The result of these computations is that every sectional curvature of the metric ds at a point $(x, r) \in T \times (r_T, \epsilon)$ is a convex combination of the two numbers

$$-\frac{(f_T'(r))^2}{(f_T(r))^2} \qquad \text{and} \qquad -\frac{f_T''(r)}{f_T(r)},$$

both of which are negative due to the properties of the function f_T. This finishes the first part.

The second part is based on a theorem of Berestovskii ([4], Theorem II.3.14) which asserts that a so-called *hyperbolic cone* over a CAT(1) space is a CAT(-1) space. Now, the shortest closed geodesic in a manifold $T \in \mathcal{T}$ has length L_T. Rescale the Riemannian metric ds_T of this manifold to the metric $\frac{2\pi}{L_T} ds_T$ in which the shortest closed geodesic in T has length 2π. In this new metric every geodesic triangle $\triangle$ in T of perimeter not greater than 2π lifts to a geodesic triangle in the universal cover $\widetilde{T} = \mathbb{E}^{n-1}$ unless $\triangle$ is a closed geodesic of length 2π. In the first case $\triangle$ satisfies the CAT(0) condition and in the second case $\triangle$ is isometric to a geodesic triangle in $\mathbb{S}^2$ contained in a great circle. The conclusion is that T with the rescaled metric is a CAT(1) space, and that the hyperbolic cone over this T is a CAT(-1) space. Since a convex open neighborhood of the apex of this hyperbolic cone is isometric to a convex open neighborhood of the point $p_T \in N$ ([14], pages 9-11), the proof of (1) is complete.

To justify (2), note first that the existence of an isometry $T_1 \to T_2$ ensures that the numbers r_{T_1} and r_{T_2} are equal. Thus we can define a map $\widehat{\rho} : C(T_1) \to C(T_2)$ by the equations

$$\widehat{\rho}(t, r) = (\rho(t), r) \qquad \text{for all } t \in T_1 \text{ and } r \in (r_{T_1}, 0] = (r_{T_2}, 0]$$

and $\widehat{\rho}(p_{T_1}) = p_{T_2}$. For every $r \in (r_{T_1}, 0] = (r_{T_2}, 0]$ the map $\widehat{\rho}$ induces an isometry between the sections $T_1 \times \{r\} \subset C(T_1)$ and $T_2 \times \{r\} \subset C(T_2)$. Also, for every $t \in T_1$ the map $\widehat{\rho}$ induces an isometry between the rays $\{t\} \times (r_{T_1}, 0]$ and $\{\rho(t)\} \times (r_{T_2}, 0]$. Hence the radial nature of the Riemannian metric $(*)$ proves that $\widehat{\rho}$ maps the set $T_1 \times (r_{T_1}, 0]$ isometrically onto the set $T_2 \times (r_{T_2}, 0]$ and so $\widehat{\rho}$ is an isometry of the completions of these two sets, which are $C(T_1)$ and $C(T_2)$, respectively. □

The cones $C(T)$ attached to the boundary ∂M are contractible, hence the fundamental group $\overline{H} = \pi_1(N)$ is obtained from the fundamental group $\overline{W} = \pi_1(M)$ by killing all elements of $\overline{W}$ represented by a loop in some boundary component $T \subset \partial M$. In other words, the inclusion $M \hookrightarrow N$ induces a surjection $\overline{W} \to \overline{H}$. Moreover, by Proposition 3.3(1) the group $\overline{H}$ is Gromov-hyperbolic. It is also non-elementary, because it has cohomological dimension n as the fundamental group of a compact orientable n-dimensional pseudomanifold. To sum up, we have constructed a non-elementary Gromov-hyperbolic quotient of $\overline{W}$.

3.3 Lifting virtual quotients

We already know that W *virtually* has a non-elementary Gromov-hyperbolic quotient. But we need to lift the surjective homomorphism $\overline{W} \to \overline{H}$ to a homomorphism of the group W itself onto a non-elementary Gromov-hyperbolic group.

Proof of Theorem 1.3. The inclusion $M \hookrightarrow N$ can be lifted to a map of the universal covers $\Psi : \widetilde{M} \to \widetilde{N}$. Note that $\widetilde{M}$ is just $\mathbb{H}^n$ with a collection of pairwise disjoint open horoballs removed. The symbols π_M and π_N will denote the projections $\widetilde{M} \to M$ and $\widetilde{N} \to N$, respectively.

The projection π_N restricted to the preimage $\pi_N^{-1}(M)$ is a covering map over M. Similarly, the projection π_N restricted to the preimage $\pi_N^{-1}(N \setminus M)$ is a covering map over $N \setminus M$. However, $N \setminus M$ is the union of the interiors of the cones $C(T)$ which are contractible and thus simply-connected. Hence the preimage $\pi_N^{-1}(M \setminus N)$ consists of connected components, each of which is isometric to the interior of a cone $C(T)$ for some $T \in \mathcal{T}$. It follows that the universal cover $\widetilde{N} = \pi_N^{-1}(M) \cup \pi_N^{-1}(N \setminus M)$ is actually the disjoint union of $\pi_N^{-1}(M)$ and sets isometric to the interiors of the cones $\{C(T)\}_{T \in \mathcal{T}}$.

Consider now any path γ joining two points of $\pi_N^{-1}(M)$. Some parts of this path are contained in sets isometric to the interiors of the cones. In a cone over a connected space every path joining two points of the base can be entirely homotoped (relative to the endpoints) to the base. Consequently, the path γ can be homotoped relative to its endpoints to a path entirely contained in $\pi_N^{-1}(M)$, which proves that the preimage $\pi_N^{-1}(M)$ is connected.

But the map $\Psi : \widetilde{M} \to \widetilde{N}$ covers the inclusion $M \hookrightarrow N$, and so the image of Ψ is contained in $\pi_N^{-1}(M)$. Moreover every path β in the connected set $\pi_N^{-1}(M)$ can be projected down to M via the map π_N, and then interpreted as a path in M it can be lifted to a path in $\widetilde{M}$ which is mapped to β by Ψ. This means that the map $\Psi : \widetilde{M} \to \pi_N^{-1}(M)$ is surjective.

The finite group $F = W/\overline{W}$ acts by isometries on $M = \widetilde{M}/\overline{W}$. On the other hand, $\widetilde{M}$ as a subset of $\mathbb{H}^n$ is invariant under the action of W on $\mathbb{H}^n$. Therefore W might be thought of as the group of those isometries of $\widetilde{M}$ that are lifts of isometries of M belonging to F.

By Proposition 3.3(2) every element $f \in F$ also acts as an isometry of N, so it can be lifted to an isometry $\widetilde{f}$ of $\widetilde{N}$. Of course, such a lift is not unique and all possible lifts $\widetilde{f}$ form an $\overline{H}$-coset in $\mathrm{Isom}(\widetilde{N})$. Let H be the group of all isometries of $\widetilde{N}$ that are lifts of some isometry $f \in F$ of N. The action of F on M is effective (i.e. a non-trivial element of F cannot act as the identity map of M). Thus the action of F on N is also effective, and we see that the quotient $H/\overline{H}$ is naturally isomorphic to F. Hence H contains a non-elementary Gromov-hyperbolic subgroup $\overline{H}$ of finite index and thus H is itself non-elementary Gromov-hyperbolic. All that is left to show is that there exists a surjective homomorphism $\eta : W \to H$.

To this end, consider $w \in W$ as an isometry of $\widetilde{M}$. We wish to define $\eta(w) \in H$. For this purpose note that $f = w\overline{W} \in F$ is an isometry of M, and therefore also an isometry of N, sending $\pi_M(p)$ to $\pi_M(w(p))$ for any point $p \in \widetilde{M}$. Moreover, since the map $\Psi : \widetilde{M} \to \widetilde{N}$ covers the inclusion $M \hookrightarrow N$, we have $\pi_N(\Psi(p)) = \pi_M(p) \in N$ and $\pi_N(\Psi(w(p))) = \pi_M(w(p)) \in N$. Hence, for a fixed point $p \in \widetilde{M}$, the isometry f of N can be lifted to the unique isometry $\widetilde{f}$ of $\widetilde{N}$ satisfying

$$\widetilde{f}(\Psi(p)) = \Psi(w(p)).$$

Observe now that the lift $\widetilde{f}$ in fact does not depend on p. This is because $\Psi(p)$ and $\Psi(w(p))$ in the above equality vary continuously with p and for any $x \in \widetilde{N}$ the set of all values $\widetilde{f}(x)$ for all lifts $\widetilde{f}$ of f is discrete in $\widetilde{N}$. Hence if we let $\eta(w) = \widetilde{f}$, then we obtain a function $\eta : W \to H$ such that

$$(\diamond) \qquad \eta(w)(\Psi(m)) = \Psi(w(m))$$

for all $w \in W$ and $m \in \widetilde{M}$. Thus for any $w_1, w_2 \in W$ we have

$$\eta(w_1 w_2)(\Psi(m)) = \Psi(w_1(w_2(m))) = \eta(w_1)(\Psi(w_2(m))) = \eta(w_1)\eta(w_2)(\Psi(m)),$$

and since $\eta(w_1 w_2)$ and $\eta(w_1)\eta(w_2)$ are lifts of the same isometry of M, namely the isometry $w_1 w_2 \overline{W} = w_1\overline{W} \cdot w_2\overline{W} \in F$, we conclude that these two lifts coincide, i.e. $\eta(w_1 w_2) = \eta(w_1)\eta(w_2)$ and η is a homomorphism.

It remains to verify that η is surjective. For this purpose take an isometry $\xi \in H$ and a point $q \in \pi_N^{-1}(M)$. Then ξ is a lift of an isometry $f \in F$ of N (hence also of M) sending $\pi_N(q) \in M$ to $\pi_N(\xi(q)) \in M$. Note that $\xi(q) \in \pi_N^{-1}(M)$, hence there exist points $p, p' \in \widetilde{M}$ such that $\Psi(p) = q$ and $\Psi(p') = \xi(q)$. In this situation we have $\xi(\Psi(p)) = \Psi(p')$ and — in view of the equality $(\diamond)$ — an element $w \in W$ sent to $\xi \in H$ by η can be defined as the lift of f satisfying $w(p) = p'$. Such a lift exists because $\pi_M(p) = \pi_N(\Psi(p)) = \pi_N(q)$ and $\pi_M(p') = \pi_N(\Psi(p')) = \pi_N(\xi(q)) = f(\pi_N(q))$.

The proof is now finished. □

Chapter 4

Reflection length in non-affine Coxeter groups

Our goal now is to prove Theorem 1.4.

We first explain why, in order to demonstrate that reflection length is unbounded in *all* non-affine Coxeter groups, we may specialize to *minimal* non-affine groups. The possibility of such specialization follows from a result of Dyer ([8], Theorem 1.1). It states that, for any element w of a Coxeter group (W, S) and any product representation $w = s_1 s_2 \ldots s_n$ $(s_i \in S)$ of minimal possible length, the reflection length of w is equal to the minimal number k for which there exist indices $1 \leq i_1 < \ldots < i_k \leq n$ such that the shorter product $s_1 \ldots \widehat{s_{i_1}} \ldots \widehat{s_{i_k}} \ldots s_n$ (where each hat denotes the omission of a factor) is equal to the identity element of W.

This result implies that for any special subgroup (W', S') of a Coxeter group (W, S) the reflection length $||\cdot||_R$ in the whole group W restricted to the subgroup W' coincides with the reflection length $||\cdot||_{R'}$ defined for the system (W', S') considered as a Coxeter group in its own right. Indeed, a standard fact about Coxeter groups (see [6]) says that for any element $w \in W'$ and any product representation $w = s_1 s_2 \ldots s_n$ $(s_i \in S)$ of minimal possible length all the generators $s_1, s_2, \ldots, s_n$ necessarily belong to S'. Thus the minimal representations of w with respect to the generating sets S and S' coincide, which in view of Dyer's theorem implies that $||w||_R = ||w||_{R'}$.

Hence if the reflection length $||\cdot||_{R'}$ is unbounded, then the reflection length $||\cdot||_R$ is unbounded, too. It remains to note that every non-affine Coxeter group W has a minimal non-affine special subgroup. Indeed, it suffices to consider all non-affine special subgroups of W and take a subgroup that is minimal with respect to inclusion. Therefore Theorem 1.4 will follow if we show that the reflection length is an unbounded function on any minimal non-affine Coxeter group.

To this end, we are going to apply quasi-morphisms. A *quasi-morphism* on a group G is a function $\varphi : G \to \mathbb{R}$ such that

$$\sup_{g,h \in G} |\varphi(gh) - \varphi(g) - \varphi(h)| < \infty.$$

Evidently, any bounded function is a quasi-morphism, and any homomorphism $G \to \mathbb{R}$ is a quasi-morphism. Since Coxeter groups are generated by elements of finite order, they do not admit any non-zero homomorphisms to $\mathbb{R}$. Thus there is no obvious candidate for an unbounded quasi-morphism on a Coxeter group.

The following fact is an elementary application of cohomology and bounded cohomology (see [9]) to Gromov-hyperbolic groups. The *bounded cohomology* $H_b^*(G;\mathbb{R})$ of a discrete group G is the cohomology of the complex $\{C_b^*(G;\mathbb{R}),\delta\}$, where for every $k \geq 1$ the space of bounded k-cochains $C_b^k(G;\mathbb{R})$ is the set of all bounded functions $\varphi : G^k \to \mathbb{R}$ and the coboundary operator $\delta : C_b^k(G;\mathbb{R}) \to C_b^{k+1}(G;\mathbb{R})$ is defined as

$$\delta\varphi(g_0,\dots,g_k) = \varphi(g_1,\dots,g_k) + \sum_{i=1}^{k}(-1)^i\varphi(g_0,\dots,g_{i-2},g_{i-1}g_i,g_{i+1},\dots,g_k) + \\ + (-1)^{k+1}\varphi(g_0,\dots,g_{k-1}).$$

Proposition 4.1. *For any non-elementary Gromov-hyperbolic group G there exists an unbounded quasi-morphism $\varphi : G \to \mathbb{R}$.*

Proof. The bounded cohomology $H_b^2(G;\mathbb{R})$ is infinite-dimensional ([9], Theorem 1.1). On the other hand, $H^2(G;\mathbb{R})$ is finite-dimensional since Gromov-hyperbolic groups are finitely presented. Hence we can consider a representative ψ of a non-trivial element of the kernel of the natural map $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$. This representative may be assumed to be a bounded function $\psi : G \times G \to \mathbb{R}$. As ψ represents the trivial element of $H^2(G;\mathbb{R})$, we have $\psi = \delta\varphi$ for some function $\varphi : G \to \mathbb{R}$, where the equality $\psi = \delta\varphi$ means just that

$$(*) \qquad \psi(g,h) = \varphi(g) - \varphi(gh) + \varphi(h) \qquad \text{for all } g,h \in G.$$

The function φ cannot be bounded. Otherwise φ would be a bounded 1-cochain and the coboundary $\psi = \delta\varphi$ would represent the zero element of the bounded cohomology $H_b^2(G;\mathbb{R})$. On the other hand, the function $\psi : G \times G \to \mathbb{R}$ is bounded and thus the equality $(*)$ shows that φ is our desired quasi-morphism. □

For a group G generated by a finite set X we can define a bi-invariant word metric on G in exactly the same way as the reflection length on a Coxeter group. Namely, let $X^* = \{gxg^{-1} : x \in X, g \in G\}$ and define

$$||g||_{X^*} = \min\{n : g = g_1g_2\dots g_n \text{ for some } g_i \in X^*\}.$$

The reflection length on a Coxeter group is then an example of a bi-invariant metric. The word 'bi-invariant' is used here to indicate that, unlike the usual word metric $||\cdot||_X$, the metric $||\cdot||_{X^*}$ is invariant under conjugation:

$$||kgk^{-1}||_{X^*} = ||g||_{X^*} \qquad \text{for all } k,g \in G.$$

Suppose that $f : G_1 \to G_2$ is a surjective homomorphism and that the group G_1 is generated by a finite set X_1. Then the group G_2 is generated by the finite set $X_2 = f(X_1)$ and the corresponding bi-invariant word metrics satisfy the obvious inequality

$$||f(g)||_{X_2^*} \leq ||g||_{X_1^*} \qquad \text{for every } g \in G_1.$$

In particular, if the metric $||\cdot||_{X_2^*}$ is unbounded, then so is the metric $||\cdot||_{X_1^*}$.

As observed first by Gal and Kędra ([10], Lemma 3.7), the behavior of bi-invariant metrics is related to the existence of unbounded quasi-morphisms. In our context we record the following result.

Proposition 4.2. *Let G be a group generated by a finite set X with the associated bi-invariant word metric $||\cdot||_{X^*}$. Then for any quasi-morphism $\varphi : G \to \mathbb{R}$ there exists a constant $C > 0$ such that*

$$||g||_{X^*} \geq C \cdot |\varphi(g)|$$

for every $g \in G$.

Proof. By the definition of a quasi-morphism there exists a real number D such that

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq D \qquad \text{for all } x, y \in G.$$

Pick any pair of elements $h, k \in G$ and apply the above inequality for $(x, y) = (h, kh^{-1})$ and $(x, y) = (kh^{-1}, h)$. We obtain that the values $\varphi(hkh^{-1})$ and $\varphi(k)$ both differ from $\varphi(h)+\varphi(kh^{-1})$ by at most D, and hence the values of φ on any two conjugate elements differ by at most $2D$.

Now let $g \in G$ be any element and let $n = ||g||_{X^*}$. Thus

$$g = g_1 x_1 g_1^{-1} g_2 x_2 g_2^{-1} \dots g_n x_n g_n^{-1} \qquad \text{for some } g_i \in G \text{ and } x_i \in X.$$

Performing straightforward induction we see that the value $\varphi(g)$ differs from the sum

$$\varphi(g_1 x_1 g_1^{-1}) + \varphi(g_2 x_2 g_2^{-1}) + \ldots + \varphi(g_n x_n g_n^{-1})$$

by at most $(n-1)D$, and so it differs from the sum

$$\varphi(x_1) + \varphi(x_2) + \ldots + \varphi(x_n)$$

by at most $(n-1)D + n \cdot 2D \leq 3nD$. On the other hand, the absolute value of the latter sum does not exceed nM, where $M = \max\{|\varphi(x)| : x \in X\}$. Combining the above inequalities we conclude that

$$|\varphi(g)| \leq (3D + M)n = (3D + M)||g||_{X^*}.$$

Therefore, the constant $C = \frac{1}{3D+M}$ has the required property. □

Corollary 4.3. *Suppose that a finitely generated group admits an unbounded quasi-morphism. Then any bi-invariant word metric on this group is unbounded.*

Finally, we are ready to deduce Theorem 1.4.

Proof of Theorem 1.4. Let (W, S) be a non-affine Coxeter group. Taking into account the considerations from the beginning of the chapter we may assume that W is minimal non-affine. Then Theorem 1.3 supplies a surjection of W onto a non-elementary Gromov-hyperbolic group H. Proposition 4.1 and Corollary 4.3 imply that any bi-invariant word metric on H is unbounded. As explained before Proposition 4.2, it follows that any bi-invariant word metric on W is unbounded. Since the reflection length is one of these metrics, the proof is complete. □

Chapter 5

Quotients of non-affine Coxeter groups

5.1 Conjectural picture

In this section we propose a candidate for a non-elementary Gromov-hyperbolic quotient of an arbitrary non-affine Coxeter group, and indicate problems that arise when one attempts to prove that this candidate has the required property. This conjecturally Gromov-hyperbolic quotient will be described in the language of simplices of groups.

As a motivation for the general case, we interpret the construction performed in Chapter 3 in terms of simplices of groups. However, we are not going to prove in detail that this interpretation coincides with the Gromov-hyperbolic quotient of W obtained in Chapter 3 — we will just not need a proof for the results of the present chapter.

Let (W, S) be a minimal non-affine Coxeter group and let $n = |S| - 1$. Recall from Chapter 2 that W is the fundamental group of the n-simplex of groups with the following data: the vertex set is equal to S and the local group assigned to the simplex spanned by $T \subset S$ is the special subgroup of W generated by $S \setminus T$. In our minimal non-affine case all local groups are finite except possibly at the vertices, where some local groups might be Euclidean. Then the group H is constructed by taking a 'sufficiently large' finite quotient of W and replacing each local group with its image in this quotient. Note that, by virtue of the residual finiteness of W, those local groups that were already finite are not affected.

This construction has a very natural generalization to all non-affine Coxeter groups:

Conjecture 5.1. *Let (W, S) be a non-affine Coxeter group and let $S' \subset S$ be a subset generating a minimal non-affine special subgroup of W (or a D_∞ subgroup).*

Let $\pi : W \to D$ be a surjective homomorphism onto a finite group and define a simplex of groups with the following data: the vertex set is equal to S and the local group assigned to the simplex spanned by $T \subset S'$ is equal to $\pi(W_{S \setminus T})$.

Then the fundamental group H of this simplex of groups is a non-elementary Gromov-hyperbolic quotient of W provided that the finite quotient D is 'sufficiently large'.

Observe that the above simplex of groups is clearly developable, since the local groups are subgroups of D and the maps between the local groups are just inclusions, so that the simplex admits a morphism to D injective on the local groups. (The same reason for developability already occurred in Section 2.3.)

The results of this chapter confirm Conjecture 5.1 in two special cases. One of them is the case $|S'| = 2$ and $W_{S'} = D_\infty$, and then we are able to give a complete characterization of non-affine Coxeter groups that admit a non-elementary virtually free quotient. The second case is $|S'| = 3$ for 'small' Coxeter groups, namely the groups with $|S| = 4$. In this direction we prove that any non-affine 4-generator Coxeter group with a non-affine 3-generator special subgroup and no infinite dihedral special subgroups admits a non-elementary Gromov-hyperbolic quotient that acts properly and cocompactly on a 2-dimensional CAT(-1) simplicial complex.

Of course, Conjecture 5.1 implies Conjecture 1.1, but the general case of Conjecture 5.1 seems to be quite difficult for the following reason. In order to prove that the group H in Conjecture 5.1 is Gromov-hyperbolic, one actually needs to show that the development X is a simplicial complex admitting a CAT(-1) metric — then H would act properly and cocompactly on the locally finite CAT(-1) complex X, implying the assertion. Thus one has to metrize the simplices of X, and there is an obvious way to do it. Namely, by Proposition 3.1 the minimal non-affine special subgroup $W_{S'}$ acts on a hyperbolic space with a simplex σ as a fundamental domain. Assume for the moment, with much loss of generality, that σ has no ideal vertices. Note that σ has a canonical labelling of its vertices by elements of S', and so does the complex X. Hence we may put a metric on X so that every simplex of X is isometric in a label-preserving way to the simplex σ in a hyperbolic space. The problem is to demonstrate that X with this metric is a CAT(-1) space. The usual method for this (see [4], II.5.4) is to show that the link of any vertex is CAT(1). If $|S'| \geq 4$, then these links are piecewise spherical complexes of dimension at least 2, and then proving the CAT(1) property is usually extremely complicated. Even when $|S'| = 3$ and the links are graphs, the fact that they contain no loops shorter than 2π requires some knowledge about the mutual location of the groups $\pi(W_{S\setminus T})$. For this reason Theorem 1.5 concerns only 'small' Coxeter groups for which this mutual location can be understood.

When σ has ideal vertices, the situation is even worse. Since σ has missing vertices, one needs to remove some vertices of X in order to put a metric on X compatible with that of σ. Then the resulting space might be CAT(-1) but the action of H is not cocompact. One would have to generalize the method from Section 3.2, where the 'horospheres' would become more complicated spaces with flat parts but also a lot of branching. Thus, Conjecture 5.1 in full generality appears to be rather challenging.

5.2 Virtually free quotients

This section is intended to give a complete description of Coxeter groups admitting a quotient that contains a non-abelian free group as a subgroup of finite index.

Proposition 5.2 (after [12]). *Let (W, S) be a non-affine Coxeter group. Then some quotient of W is a non-elementary virtually free group if and only if W contains an infinite dihedral D_∞ special subgroup that does not split as a direct factor.*

The condition of not splitting as a direct factor can be stated more precisely by saying that this D_∞ special subgroup (generated by 2 elements of S) does not commute with the special subgroup generated by the remaining elements of S.

Proof. The 'only if' direction is easy and well-known. It is based on the facts that a virtually free group acts on a tree with finite vertex stabilizers (this follows from Bass-Serre theory, see [6], Section E.1) and that a Coxeter group with $m_{st} < \infty$ for all $s, t \in S$ has no action on a tree without a global fixed point ([6], Example E.1.12). Thus one obtains a D_∞ special subgroup, and the assumption that the quotient is not virtually cyclic forces at least one such subgroup not to split as a direct factor.

The 'if' direction is more interesting and this is where the ideas of [12] have to be completed and turned into a rigorous proof.

Let $\{s, t\}$ be a subset of S generating a D_∞ special subgroup that does not split as a direct factor. Hence we may assume that there exists a generator $u \in S \setminus \{s, t\}$ such that $m_{su} > 2$. Let $W_1 = W_{S\setminus\{s\}}$, $W_2 = W_{S\setminus\{t\}}$ and $W' = W_1 \cap W_2 = W_{S\setminus\{s,t\}}$ so that there is an amalgamated free product decomposition $W = W_1 *_{W'} W_2$.

The cosets W', sW' and $susW'$ of the subgroup $W' \subset W_2$ are all different. This is because the elements s and sus are different, do not belong to W' (since $sus \neq u$), and $s^{-1}(sus) = us \notin W'$. Therefore the index $[W_2 : W']$ is at least 3. Also, $[W_1 : W'] \geq 2$.

We claim that there exists a homomorphism $\alpha : W \to D$ onto a finite group such that $[\alpha(W_1) : \alpha(W')] \geq 2$ and $[\alpha(W_2) : \alpha(W')] \geq 3$. Note that this does not follow from the residual finiteness of W itself. What we need here is the notion of a *separable subgroup*. A subgroup $K \subset G$ is separable in G if for any $g \in G \setminus K$ there exists a homomorphism $f : G \to D$, where D is a finite group, such that $f(g) \notin f(K)$. (This is a strong assumption: there exist non-separable finitely generated subgroups of $\mathbb{F}_2 \times \mathbb{F}_2$.)

It is a result of Cooper, Long and Reid ([5], Theorem 1.4) that special subgroups of a Coxeter group are separable. Thus there are homomorphisms $\alpha_i : W \to D_i$ $(1 \leq i \leq 4)$ onto finite groups with the following properties: $\alpha_1(s) \notin \alpha_1(W')$, $\alpha_2(sus) \notin \alpha_2(W')$, $\alpha_3(s^{-1}(sus)) \notin \alpha_3(W')$, $\alpha_4(t) \notin \alpha_4(W')$. The product $\alpha = \prod_{i=1}^4 \alpha_i : W \to \prod_{i=1}^4 D_i$ then satisfies the same four relations:

$$\alpha(s) \notin \alpha(W'), \quad \alpha(sus) \notin \alpha(W'), \quad \alpha(s^{-1}(sus)) \notin \alpha(W'), \quad \alpha(t) \notin \alpha(W').$$

The last relation shows that $\alpha(W') \neq \alpha(W_1)$ and the first three relations imply that $\alpha(W')$, $s\alpha(W')$ and $sus\alpha(W')$ are distinct cosets of the subgroup $\alpha(W') \subset \alpha(W_2)$. Consequently, $[\alpha(W_1) : \alpha(W')] \geq 2$ and $[\alpha(W_2) : \alpha(W')] \geq 3$, as claimed.

Finally, observe that the group $W = W_1 *_{W'} W_2$ admits a homomorphism onto the virtually free group $A = \alpha(W_1) *_{\alpha(W')} \alpha(W_2)$. The group A is non-elementary since the inequalities $[\alpha(W_1) : \alpha(W')] \geq 2$ and $[\alpha(W_2) : \alpha(W')] \geq 3$ imply that the locally finite Bass-Serre tree associated to the decomposition of A is not a line. □

There is a remarkable rigidity phenomenon here. Namely, by a result of Margulis and Vinberg (see [6], Theorem 14.1.2) every non-affine Coxeter group W has a subgroup $\overline{W}$ of finite index that admits a non-abelian free quotient $\overline{U}$. But Proposition 5.2 implies that the homomorphism $\overline{W} \to \overline{U}$ can be lifted to a homomorphism from W onto a virtually free group only in specific cases. This underlines the importance of imposing the equivariance requirement in Section 3.2 which is later necessary in Section 3.3.

5.3 Quotients of 4-generator groups

The purpose of this section is to prove Theorem 1.5.

We first deal with two exceptional cases. If (W, S) a 4-generator non-affine Coxeter group such that $m_{st} = \infty$ for some $s, t \in S$, then the special subgroup $W_{\{s,t\}} = D_\infty$ does not split as a direct factor, since the product $D_\infty \times W_{S\setminus\{s,t\}}$ would be affine as the second factor is a dihedral group. Therefore Proposition 5.2 provides a non-elementary virtually free quotient of W. Thus we may assume that $m_{st} < \infty$ for all $s, t \in S$. Moreover, if all 3-generator special subgroups of W are affine, then W is a minimal non-affine Coxeter group and Theorem 1.3 can be applied. For this reason we add a second assumption: some 3-generator special subgroup of W is non-affine. It follows from the condition $m_{st} < \infty$ for all $s, t \in S$ that this special subgroup is a cocompact *hyperbolic triangular group*, i.e. a group acting by isometries on $\mathbb{H}^2$ with fundamental domain a compact triangle such that the 3 generators act as reflections across the sides of this triangle. (This is a special case of Proposition 3.1).

Let us now fix some notation. Let (W, S) be a non-affine Coxeter group, where $S = \{s_1, s_2, s_3, s_4\}$. We assume that $m_{s_i s_j} < \infty$ for all $1 \leq i, j \leq 4$. Suppose also that the special subgroup $W_{\{s_1,s_2,s_3\}}$ is a hyperbolic triangular group.

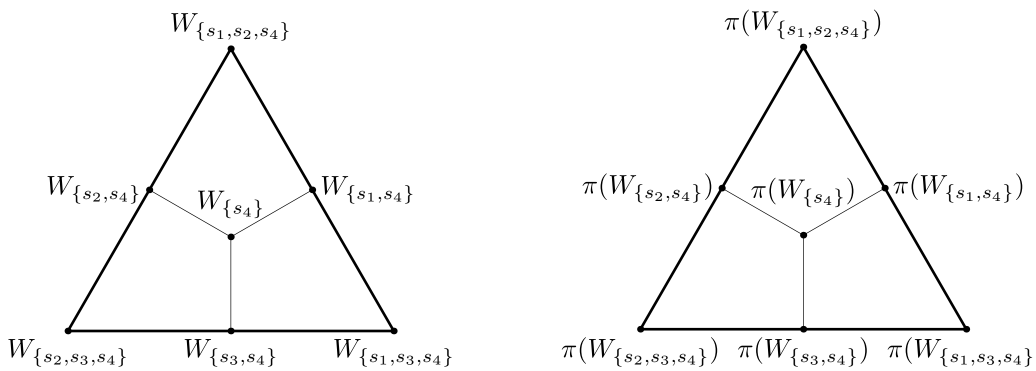
Of course, many groups W satisfying the above conditions are Gromov-hyperbolic themselves. In fact, this happens exactly when every 3-generator special subgroup of W is finite or hyperbolic triangular (see [6], Corollary 12.6.3). But one can easily construct examples with a Euclidean 3-generator subgroup, and then the existence of a non-elementary Gromov-hyperbolic quotient of W does not follow from any of the results that we have already proved.

In order to be ready for the proof of Theorem 1.5 we have to review the behavior of curvature in 2-dimensional simplicial complexes. Suppose that X is a connected 2-dimensional simplicial complex with a metric so that each triangle is isometric to a compact triangle in $\mathbb{H}^2$. Then, under some mild assumptions (for example that there are finitely many isometry types of triangles in X, see [4], I.7.19) X is a geodesic space. Moreover, X is locally CAT(-1) at all points except possibly at the vertices. There is a well-known *link condition* which allows to detect exactly when X is locally CAT(-1) at a vertex (see [4], II.5.4). In our situation this condition says that the link of a vertex, which is a graph, should not contain loops shorter than 2π. The link $\mathrm{lk}(v)$ of a vertex v is constructed as follows: the vertices of $\mathrm{lk}(v)$ are the edges of X originating at v, the edges of $\mathrm{lk}(v)$ are the triangles of X containing v, and the incidence relation in $\mathrm{lk}(v)$

is naturally inherited from the incidence relation in X. (One may think of $\mathrm{lk}(v)$ as a metric sphere in X of very small radius, centered at v.) Moreover, the length of an edge $\mathrm{lk}(v)$ is equal to the angle of the corresponding triangle in X at the vertex v. If the link condition is satisfied at all vertices, then X is negatively curved. If in addition X is simply-connected — for example if it arises as the development of a triangle of groups — then X is a CAT(-1) space (see [4], II.4.1(2)).

Proof of Theorem 1.5. We remind the reader that the assumptions made in the paragraph starting with 'Let us now fix some notation' are still in force. We divide the proof into several steps.

Step 1. Consider the following two triangles of groups:



The triangle of groups on the left side has finite local groups assigned to the whole triangle and to the edges (the latter follows from the assumption $m_{s_is_j} < \infty$). The local groups assigned to the vertices might be infinite, however.

Now let $\pi : W \to D$ be a homomorphism onto a sufficiently large finite group, where the meaning of 'sufficiently large' will be developed throughout the whole proof by adding successively new conditions that the map π should satisfy. The only property of W that we will use in this process is the residual finiteness. Observe that for a finite subgroup $K \subset W$ we can force π to map K isomorphically onto the image $\pi(K)$. Indeed, one can ensure that any element of the finite set $\{gk^{-1} : g, k \in K \text{ and } g \neq k\}$ has non-trivial image in D, and then $\pi(g) \neq \pi(k)$ for any pair $g, k \in K$ with $g \neq k$, so that π is injective on K. Thus a prescribed finite collection of finite subgroups of W can be mapped isomorphically by π onto their images, provided that π is chosen appropriately.

In the triangle of groups on the right side all local groups are finite. Moreover, we demand that every finite local group on the left side should be mapped isomorphically onto the corresponding group on the right side.

As before, both triangles of groups are developable since each of them is built out of subgroups of a fixed group. Let X and Y denote the developments of the triangle on the left side and on the right side, respectively. Our ultimate goal is to show that the fundamental group of the triangle on the right side is a required Gromov-hyperbolic quotient of W.

Step 2. We now wish to put a metric on the simplical complexes X any Y. This is done in the way already indicated in Section 5.1. Recall that $W_{\{s_1,s_2,s_3\}}$ is a hyperbolic triangular group and so it acts by isometries on $\mathbb{H}^2$ with a triangle $\triangle$ as a fundamental domain. The triangle $\triangle$ has a canonical labelling of the vertices by elements of the set $\{s_1, s_2, s_3\}$. So do the complexes X and Y: a vertex of the fundamental triangle is labelled by s_i ($i \in \{1,2,3\}$) if the special subgroup (or its image under π) assigned to this vertex does *not* involve the generator s_i. Obviously, this labelling can be extended to the whole of X and Y by using the group action. Then every triangle of X and Y has three distinct vertex labels, and there exists a unique metric on such a triangle so that there is a label-preserving isometry from this triangle onto $\triangle$.

In this way we turn X and Y into geodesic metric spaces which are locally CAT(-1) at every point except possibly at the vertices.

Step 3. We will describe an algebraic method of veryfing the link condition in the complexes X and Y.

For illustration, and without any loss of generality, take the vertex p of the fundamental triangle of X stabilized by the group $V = W_{\{s_1,s_2,s_4\}}$. Let also $E_1 = W_{\{s_1,s_4\}}$, $E_2 = W_{\{s_2,s_4\}}$ and $F = E_1 \cap E_2 = W_{\{s_4\}}$. Then the triangles in X containing p are in bijective correspondence with the F-cosets contained in V, and similarly every edge in X containing p can be identified with some E_1-coset or E_2-coset contained in V.

A loop in the link lk(p) can be now described as a sequence of triangles containing p such that any two neighboring triangles (and also the last and the first triangle) share a common edge, but no three consecutive triangles share an edge. Sequences of triangles of this type correspond to sequences of F-cosets v_1F, v_2F, $\ldots$, v_mF (with $v_{m+1}F = v_1F$) such that for $i = 1, 2, \ldots, m$ the F-cosets v_iF and $v_{i+1}F$ are contained in a common E_1-coset or a common E_2-coset. But this condition is equivalent to $v_i^{-1}v_{i+1} \in E_1$ or $v_i^{-1}v_{i+1} \in E_2$, respectively. Moreover, two consecutive elements $v_i^{-1}v_{i+1}$ and $v_{i+1}^{-1}v_{i+2}$ cannot both belong to E_1 and cannot both belong to E_2. Otherwise v_iF, $v_{i+1}F$ and $v_{i+2}F$ would be contained in a E_1-coset or in a E_2-coset, so the sequence of triangles would contain three consecutive triangles sharing an edge, which is not allowed.

Now if we let $z_i = v_i^{-1}v_{i+1}$ for $i = 1, 2, \ldots, m$, then the equality $z_1z_2\ldots z_m = 1$ is satisfied in V. Each of the elements z_1, z_2, $\ldots$, z_m, z_1 belongs to one of the groups E_1 or E_2, but no two consecutive elements belong to the same group. It follows that these elements alternately belong to $E_1 \setminus E_2$ and $E_2 \setminus E_1$. (In particular m is even.)

One readily sees that the converse is also true: whenever there is an equality

$$(*) \qquad a_1b_1a_2b_2\ldots a_kb_k = 1 \qquad \text{for some } a_i \in E_1 \setminus E_2 \text{ and } b_i \in E_2 \setminus E_1,$$

then there is also a loop in lk(p) consisting of $2k$ edges, namely the loop

$$F, \quad a_1F, \quad a_1b_1F, \quad a_1b_1a_2F, \quad a_1b_1a_2b_2F, \quad \ldots, \quad a_1b_1a_2b_2\ldots a_{k-1}b_{k-1}a_kF.$$

The conclusion is that the minimal number of edges of a loop in lk(p) is equal to $2k$, where k is a minimal positive integer for which the condition $(*)$ holds. In this setting one usually says that the *angle* between the subgroups $E_1, E_2 \subset V$ is equal to $\frac{\pi}{k}$.

Step 4. We would like to verify that the link condition is satisfied at every vertex of X. The group action reduces this to showing the condition at a vertex of the fundamental triangle. We will work with the vertex p stabilized by $W_{\{s_1,s_2,s_4\}}$; the proof for the two other vertices is entirely analogous.

A result of Barnhill ([1], Proposition 6.8) implies that the angle between the special subgroups $W_{\{s_1,s_4\}}$ and $W_{\{s_2,s_4\}}$ of the Coxeter group $W_{\{s_1,s_2,s_4\}}$ is equal to $\frac{\pi}{m_{s_1s_2}}$. Thus every loop in $\mathrm{lk}(p)$ contains at least $2m_{s_1s_2}$ edges.

But the fundamental triangle of X is isometric to the triangle $\triangle \subset \mathbb{H}^2$ associated with the hyperbolic triangular group $W_{\{s_1,s_2,s_3\}}$. The angle of $\triangle$ at the vertex corresponding to p equals $\frac{\pi}{m_{s_1s_2}}$. Hence every edge in the link $\mathrm{lk}(p)$ has length $\frac{\pi}{m_{s_1s_2}}$ and every loop in this link has length at least $2m_{s_1s_2} \cdot \frac{\pi}{m_{s_1s_2}} = 2\pi$, which is exactly the link condition in X that we intended to prove.

Step 5. We are now going to prove that, for an appropriate choice of the homomorphism π, the complex Y also satisfies the link condition.

Examining carefully the proof of the link condition in X and recalling that triangles of X are isometric in a label-preserving way to triangles of Y we arrive at the following statement: the verification of the link condition in Y amounts to checking that the minimal number of edges of a loop in the link of a vertex in Y is the same as in X. To be more precise, take again the vertex $p \in Y$ stabilized by the group $\pi(W_{\{s_1,s_2,s_4\}})$. Then the link condition at the vertex p is equivalent to the fact that the angle between the finite subgroups $\pi(W_{\{s_1,s_4\}}) \cong W_{\{s_1,s_4\}}$ and $\pi(W_{\{s_2,s_4\}}) \cong W_{\{s_2,s_4\}}$ of the group $\pi(W_{\{s_1,s_2,s_4\}})$ equals at most $\frac{\pi}{m_{s_1s_2}}$.

Let $C_1 = \pi(E_1) \cong E_1$, $C_2 = \pi(E_2) \cong E_2$ and $U = \pi(V)$, where the groups E_1, E_2 and V are defined as in Step 3. According to the final part of Step 3 our task is to show that if

$$(\diamond) \qquad a_1b_1a_2b_2 \ldots a_kb_k = 1 \qquad \text{for some } a_i \in C_1 \setminus C_2 \text{ and } b_i \in C_2 \setminus C_1$$

in the group D, then $k \geq m_{s_1s_2}$. In other words, we need to demonstrate that, for a suitable homomorphism π, the equality $(\diamond)$ cannot hold for $k < m_{s_1s_2}$.

However, any expression of the form $a_1b_1a_2b_2 \ldots a_kb_k$, where $a_i \in E_1 \setminus E_2$, $b_i \in E_2 \setminus E_1$ and $k < m_{s_1s_2}$, represents a non-trivial element of the group V. This is a direct consequence of the fact cited in Step 4 that the angle between the subgroups E_1 and E_2 of the group V is equal to $\frac{\pi}{m_{s_1s_2}}$. Moreover, the set Q of all such expressions is finite since the groups E_1 and E_2 are finite. Therefore we can choose π that maps each element of Q to a non-trivial element of D. Since the sets E_1, E_2 are mapped by π bijectively onto the sets C_1, C_2, respectively, we infer that every element of D that can be represented as in the left-hand side of $(\diamond)$ for $k < m_{s_1s_2}$ belongs to the image $\pi(Q)$, and hence is non-trivial. Consequently, the link condition at the vertex p holds for any homomorphism π satisfying the above requirement.

Putting together this and the two analogous requirements coming from the two other label types in Y we obtain a homomorphism π such that the complex Y satisfies the link condition everywhere, and thus is a CAT(-1) space.

Step 6. As explained in Section 2.3, the fundamental group of the triangle of groups from which the complex X was developed is just the group W. Let H be the fundamental group of the triangle of groups related to the complex Y. All local groups of the latter triangle are quotients of the respective local groups of the former triangle, in a way compatible with the inclusion maps. This implies that H is a quotient of W.

The vertex stabilizers of the action of H on the complex Y are finite. Hence the action is proper, and obviously cocompact since the quotient of the action is a triangle. Thus the group H acts properly and cocompactly on a CAT(-1) space so that it is Gromov-hyperbolic.

It remains to argue that H is non-elementary. But it cannot be finite, since a finite group acting on a CAT(-1) space has a fixed point (see [4], II.2.8(1)). The action of H on Y, on the other hand, has no fixed points in the fundamental triangle (or anywhere else), because no local subgroup of the related triangle of groups contains all other local groups. Also, H cannot be virtually cyclic due to the assumption $m_{s_i s_j} < \infty$ for all $1 \leq i, j \leq 4$; see the first part of the proof of Proposition 5.2. Hence H is indeed non-elementary, and the proof is complete. □

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